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On a Harnack–Natanzon theorem for the family of real forms of Riemann surfaces

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Abstract

An old theorem of Harnack states that a symmetry of a compact Riemann surface X of genus g, ($g \geq 2$) has at most $g+1$ disjoint simple closed curves of fixed points, each of which is called the *oval of* X. Much more recently Natanzon proved that for $\nu(g)$ being the maximum number of ovals that a surface of genus g admits, $\nu(g) \leq 42(g-1)$. We show in this paper that actually for $g \neq 2,3,5,7,9$, $\nu(g) \leq 12(g-1)$, that this bound is sharp for infinitely many g and we calculate $\nu(g)$ for the mentioned above exceptional values of g as well. ©1997 Elsevier Science B.V.

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1. Introduction

Let X be a compact Riemann surface of genus $g \geq 2$. A *symmetry* of X is an antiholomorphic involution σ, i.e., an orientation-reversing automorphism of X of order 2. A surface admitting a symmetry is said to be *symmetric*. An old theorem of Harnack [9] states that the set $F(\sigma)$ of fixed points of σ is either empty or it consists of $\|\sigma\| \leq g+1$ disjoint simple closed curves to which we shall refer as to the *ovals of* σ. A simple closed curve on a Riemann surface X will be said to be an *oval on* X if it is an oval of some symmetry of X. Let us denote by $\|X\|$ the number of ovals of X.

For every integer $g \geq 2$, let $\nu(g)$ be the maximum of $\|X\|$, where X runs over the family of compact Riemann surfaces of genus g. It is not hard to prove that a surface X_g corresponding to the equation $y^2 = x^{2(g+1)} - 1$ admits a symmetry with $g+1$ ovals and so in particular $\nu(g) \geq g+1$. Much more recently, Natanzon [15, 16]

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obtained, by topological methods, an upper bound $\nu(g) \leq 42(g-1)$. We contribute in this paper to a better knowledge of this function by proving that $\nu(g) \leq 12(g-1)$, for $g \neq 2,3,5,7,9$, that this bound is sharp for infinitely many g and we calculate $\nu(g)$ for the above-mentioned exceptional values of g as well.

Given a pair (X, σ) consisting of a Riemann surface X and a symmetry σ of X, the orbit space X/σ admits a unique structure of Klein surface such that $X \to X/\sigma$ is a morphism of Klein surfaces provided that one regards X as a Klein surface [1] (see also [17, 7, 5]). Conversely, given a Klein surface Y its canonical Riemann double cover $X = Y^+$ admits a symmetry σ such that X/σ and Y are isomorphic [1] (see also [17, 7, 5]). As a result the study of symmetries of a given Riemann surface X is equivalent to the study of Klein surfaces having X as its canonical double Riemann cover.

One of the most important problems in real algebraic geometry is the one of real moduli, i.e, real classification of real objects which in general is quite different of the real part of the corresponding complex moduli. Recent developments are due to Natanzon [12, 13], Bochnak et al. [3], Ballico [2], Seppälä and Silhol [19], Silhol [20] and a particularly interesting question is the following one studied in [14, 16, 6, 4]: given a complex irreducible curve X how many nonbirationally isomorphic real algebraic curves do exist whose complexifications are birationally isomorphic to X? Let us call them *real forms* of X. If we assume X to be a compact Riemann surface i.e, X is projective and smooth, the real forms are in bijection with the conjugacy classes in the group of automorphisms of X of symmetries with fixed points. Moreover, the number of ovals of the symmetry associated to the real form Y is the number of connected components of a projective normalization of Y.

2. Preliminaries

A Klein surface is a compact topological surface equipped with a dianalytic structure [1]. A very useful role in the study of the groups of automorphisms of such surfaces play *NEC groups*, by which we mean discrete and cocompact subgroups of the group $\mathscr{G}$ of isometries of the hyperbolic plane $\mathscr{H}$, including those which reverse the orientation. Having such a group Λ the orbit space $\mathscr{H}/\Lambda$ is a compact surface with a dianalytic structure due to the fact that the group $\mathscr{G}$ coincides with the extended Möbius group which consists of Möbius and anti-Möbius transformations (the later being Möbius transformations followed by the reflection along the imaginary axis $z \mapsto -\bar{z}$). Now the subgroup $\mathscr{G}^+$ of $\mathscr{G}$ consisting of orientation-preserving isometries has obviously index 2 and coincides with the Möbius group. An *NEC* group is called a *Fuchsian group* if it is contained in $\mathscr{G}^+$ and a *proper NEC group* otherwise.

Macbeath and Wilkie [11, 24] associated to every *NEC* group Λ a signature that has the form

$$(g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}), \tag{1}$$

and determines the algebraic structure of the group. The numbers $m_i \geq 2$ are called the *proper periods*, the brackets $(n_{i1}, \ldots, n_{is_i})$ the *period cycles*, the numbers $n_{ij} \geq 2$ the *link periods* and $g \geq 0$ is said to be the *orbit genus* of Λ.

A Fuchsian group can be regarded as an *NEC* group with signature

$$(g; +; [m_1, \ldots, m_r]; \{-\}). \tag{2}$$

If Λ is a proper *NEC* group with signature (1) then by [23], its *canonical Fuchsian subgroup* $\Lambda^+ = \Lambda \cap \mathscr{G}^+$ has signature

$$(\alpha g + k - 1; +; [m_1, m_1, \ldots, m_r, m_r, n_{11}, \ldots, n_{1s_1}, \ldots, n_{k1}, \ldots, n_{ks_k}]; \{-\}), \tag{3}$$

where $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ otherwise.

The group with signature (1) has a presentation with the following generators and relations.

$$\left\{\begin{array}{l} \textit{Generators}: \\ \quad \text{(i)}\ x_i,\ i = 1, \ldots, r, \\ \quad \text{(ii)}\ c_{ij},\ i = 1, \ldots, k,\ j = 0, \ldots, s_i, \\ \quad \text{(iii)}\ e_i,\ i = 1, \ldots, k, \\ \quad \text{(iv)}\ a_i,\ b_i,\ i = 1, \ldots, g \text{ if the sign is } +, \\ \qquad\quad d_i,\ i = 1, \ldots, g \text{ if the sign is } -, \\ \textit{Relations}: \\ \quad \text{(i)}\ x_i^{m_i} = 1,\ i = 1, \ldots, r, \\ \quad \text{(ii)}\ c_{is_i} = e_i^{-1} c_{i0} e_i,\ i = 1, \ldots, k, \\ \quad \text{(iii)}\ c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1} c_{ij})^{n_{ij}} = 1,\ i = 1, \ldots, k,\ j = 1, \ldots, s_i, \\ \quad \text{(iv)}\ x_1 \ldots x_r e_1 \ldots e_k [a_1, b_1] \ldots [a_g, b_g] = 1 \text{ if the sign is } +, \\ \qquad\quad x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_g^2 = 1 \text{ if the sign is } -. \end{array}\right. \tag{4}$$

Any system of generators of an *NEC* group Λ satisfying the above relations will be called a *canonical system* of its generators. The only orientation reversing canonical generators of Λ are all c_{ij} that represent reflections and all d_i that represent glide reflections. We will say that a reflection c_{ij} corresponds to the link periods n_{ij} and n_{ij+1}.

Every *NEC* group has associated a fundamental region, whose hyperbolic area depends only on the signature of the group and for a group with signature (1) is given by

$$\mu(\Lambda) = 2\pi \left(\alpha g + k - 2 + \sum_{i=1}^{r} (1 - 1/m_i) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} (1 - 1/n_{ij}) \right), \tag{5}$$

where α is defined as in (3).

It is known that an abstract group Γ with the presentation (4) can be realized as an *NEC* group with signature (1) if and only if the right-hand side of (5) is positive. If Γ is a subgroup of finite index in an *NEC* group Λ, then it is an *NEC* group itself and the classical Hurwitz–Riemann formula can be read as

$$[\Lambda : \Gamma] = \mu(\Gamma)/\mu(\Lambda). \tag{6}$$

For a given Klein surface X, Alling and Greenleaf [1] have constructed a certain double cover X^+ being a Riemann surface. The *algebraic genus* of X is defined as the genus of X^+ and for a surface of topological genus g having k boundary components it turns out to be equal to $p = \alpha g + k - 1$, where $\alpha = 2$ if X is orientable and $\alpha = 1$ otherwise. (If X is a Riemann surface, then its algebraic genus equals, by definition, the topological one.) The existence of such covers on the one hand and the nice behavior of quotients of Riemann surfaces under antiholomorhic involutions, mentioned before, provide a link between the categories of Riemann and Klein surfaces and permit to prove for Klein surfaces similar results to classical ones concerning of Riemann surfaces.

For example, a counterpart of the classical Riemann uniformization theorem by which any compact Klein surface X of algebraic genus $p \geq 2$ can be represented as $\mathscr{H}/\Gamma$ for some *NEC* group Γ is known [18]. Furthermore, if X has topological genus g and k boundary components, then Γ can be chosen to be a *surface NEC group,* i.e., an *NEC* group with signature

$$(g; \pm; [-]; \{(-), \overset{k}{\ldots}, (-)\}). \tag{7}$$

Surface *NEC* groups are characterized among all *NEC* groups by the fact that the reflections are their only elements of finite order. Fuchsian surfaces groups can be seen as *NEC* groups with signatures $(g; +; [-]; \{-\})$. Such groups are characterized among all *NEC* groups by the fact that they contain only orientation-preserving elements of infinite order.

Conversely, given an *NEC* surface group Γ with signature (7), the orbit space $X = \mathscr{H}/\Gamma$ is a surface of topological genus g having k boundary components admitting a unique structure of Klein surface, so that the canonical projection $\mathscr{H} \to X$ is a morphism of Klein surfaces [1]. Furthermore, its canonical Riemann double cover X^+ is $\mathscr{H}/\Gamma^+$, where Γ^+ is the canonical Fuchsian subgroup of Γ.

Now given a Riemann (respectively Klein) surface X represented as the orbit space $\mathscr{H}/\Gamma$ for some Fuchsian (respectively *NEC*) surface group Γ, a finite group G is a group of automorphisms of X if and only if $G \cong \Lambda/\Gamma$ for some *NEC* group Λ. In such a case we say that G is a *smooth* Fuchsian (resp. *NEC*) factor group of Λ.

3. General formulas

Given a symmetry σ of a Riemann surface X represented as $\mathscr{H}/\Gamma$ for a surface Fuchsian group Γ, we shall find in this section general formulas for $\|\sigma\|$ and $\|X\|$, which involve only $\mathrm{Aut}^{\pm}(X) = \Lambda/\Gamma$. We shall use these formulas in the next section to find the bound for the function ν introduced in Section 1. The following notations remain fixed throughout the rest of the paper.

Let $\mathrm{Aut}^+(X)$ be the group of the orientation-preserving automorphisms of X. Then $\mathrm{Aut}^+(X) = \Delta/\Gamma$ for some Fuchsian group Δ which is the normalizer of Γ in $\mathscr{G}^+$. Now X is symmetric if and only if there exists a proper *NEC* group Λ containing Δ as a subgroup of index 2 and Γ as a normal subgroup. In such a case $G = \Lambda/\Gamma = \mathrm{Aut}^{\pm}(X)$. Denote $G^+ = \mathrm{Aut}^+(X)$. Now a symmetry of X is an arbitrary element $\sigma \in G \setminus G^+$ of order 2. Denote by $\langle\sigma\rangle$ the group generated by σ and represent it as Γ_σ/Γ for some *NEC* subgroup Γ_σ of Λ. Then $X/\sigma \cong \mathscr{H}/\Gamma_\sigma$ and so $\|\sigma\|$ is the number of period cycles of the signature of Γ_σ. The last, however, is rather difficult to find, as in general Γ_σ is not a normal subgroup of Λ.

Observe that Λ acts as a group of permutations on the set of cosets Λ/Γ_σ in the same way as G does on $G/\langle\sigma\rangle$ (via the correspondence established by the canonical projection $\theta : \Lambda \to G$). On the other hand, Hoare [10] developed recently an algorithm to find in such a situation the proper periods and the period cycles of Γ_σ. Unfortunately, its computational complexity restricts essentially the area of theoretical and effective calculations for large indices even using modern, sophisticated, computational techniques. Below we present a direct method for finding the number of empty period cycles of Γ_σ.

In order to state our results we need further notations. Given a canonical system of generators for an *NEC* group Λ, let $\{c_i\colon i \in I\}$ be the subset of its reflections to which we shall refer as to *canonical reflections*. We introduce two equivalence relations on the set I:

- $i \sim i'$ if and only if c_i and $c_{i'}$ are conjugate in Λ and
- $i \approx i'$ if and only if $\theta(c_i)$ and $\theta(c_{i'})$ are conjugate in $\theta(\Lambda)$.

Let $I_\approx$ be a set of representatives for $I/_\approx$, and given $i \in I_\approx$ let $I(i)$ be a system of representatives for $[i]_\approx/_\sim$ containing i. Then $I_\sim = \bigcup_{i\in I_\approx} I(i)$ is a set of representatives for $I/_\sim$. Finally given $g \in G$ let $C(G,g)$ be the centralizer of g in G. With these notations we have the following result.

Theorem 3.1. *Let X be a symmetric Riemann surface uniformized by a Fuchsian surface group Γ and let $\mathrm{Aut}^{\pm}(X) = \Lambda/\Gamma$. Then a symmetry σ of X with a nonempty set of fixed points is conjugated to $\theta(c_i)$ for some canonical reflection c_i of Λ and it has*

$$k_i = \sum_{j\in I(i)} [C(\theta(\Lambda),\theta(c_j)) : \theta(C(\Lambda,c_j))]$$

ovals.

Proof. Let $\langle\sigma\rangle = \Gamma_\sigma/\Gamma$. Observe that $\Gamma_\sigma^+ = \Gamma$ and so by (3), Γ_σ has signature $(g';\pm;[-];\{(-),\overset{k}{\ldots},(-)\})$. Each period cycle of Γ_σ corresponds to an oval of σ on the one hand and to a reflection of Λ on the other. By [11], each reflection of Λ is conjugate in Λ to some of its canonical reflections c_i. So in particular σ is conjugate in $\mathrm{Aut}^{\pm}(X)$ to $\theta(c_i)$ for some $i \in I$. Thus the first part of the theorem follows. Similar arguments applied to Γ_σ show that two reflections from Γ_σ give rise to the same oval if and only if they are conjugate in Γ_σ.

Conjugate with respect to G symmetries have the same number of ovals. Hence, we can assume, without loss of generality, that $\sigma = \theta(c_i)$. If a conjugation $c_j^u = uc_ju^{-1}$ belongs to Γ_σ, then $\theta(c_j^u) = \sigma$ and so in particular $j \in [i]_\approx$. Furthermore, if $j' \sim j$ for some $j, j' \in [i]_\approx$ then for arbitrary $u' \in \Lambda$, $c_{j'}^{u'} = c_j^u$ for some $u \in \Lambda$. So in order to find the number of ovals of σ we have to count down the reflections of the form c_j^w which are in Γ_σ and are nonconjugate there, where j and w run over the sets of elements of $I(i)$ and Λ, respectively. Denote by C_i the inverse image in Λ of the centralizer of $\theta(c_i)$ in $\theta(\Lambda)$.

First observe that $c_i^w \in \Gamma_\sigma$ if and only if $w \in C_i$. In particular, C_i normalizes Γ_σ. Now given $w, w' \in C_i$, the reflections c_i^w and $c_i^{w'}$ are conjugate in Γ_σ if and only if $w^{-1}\gamma w' \in C(\Lambda, c_i)$ for some $\gamma \in \Gamma_\sigma$. On the other hand, $w^{-1}\gamma w' = (w^{-1}w')(w'^{-1}\gamma w')$. So as C_i normalizes Γ_σ, we see that $w^{-1}\gamma w' \in C(\Lambda, c_i)$ if and only if $w^{-1}w' \in C(\Lambda, c_i)\Gamma_\sigma = C(\Lambda, c_i)\Gamma$. Concluding we see that conjugates of c_i give rise to $[C_i : C(\Lambda, c_i)\Gamma]$ empty period cycles in Γ_σ.

Now let $g\theta(c_j)g^{-1} = \theta(c_i)$ and let $\theta(w) = g$. Then $wc_jw^{-1} = c_i\gamma$ and $c_j = w^{-1}c_iw\gamma'$ for some $\gamma, \gamma' \in \Gamma$. As $vc_jv^{-1} = (vw^{-1}c_iwv^{-1})(v\gamma'v^{-1})$ we see that $c_j^v \in \Gamma_\sigma$ if and only if $vw^{-1}c_iwv^{-1} \in \Gamma_\sigma$ which is equivalent to $vw^{-1} \in C_i$. We shall show that given $u, u' \in C_i$, the reflections $c_j^v, c_j^{v'} \in \Gamma_\sigma$, where $v = uw$ and $v' = u'w$ are conjugate in Γ_σ if and only if $u^{-1}u' \in w(C(\Lambda, c_i)\Gamma)w^{-1}$. Indeed c_j^v and $c_j^{v'}$ are conjugate in Γ_σ if and only if $v^{-1}\gamma v' \in C(\Lambda, c_j)$ for some $\gamma \in \Gamma_\sigma$. Now as C_i normalizes Γ_σ we have

$$v'^{-1}\gamma v' = w^{-1}(u'^{-1}\gamma u')w \in w^{-1}\Gamma_\sigma w = \Gamma_{\theta(c_j)} = \theta^{-1}(\langle\theta(c_j)\rangle)$$

by the choice of w. Since $v^{-1}\gamma v' = (v^{-1}v')(v'^{-1}\gamma v'$, c_j^v and $c_j^{v'}$ are conjugate in Γ_σ if and only $v^{-1}v' \in C(\Lambda, c_j)\Gamma_{\theta(c_j)} = C(\Lambda, c_j)\Gamma$. But then $u^{-1}u' = w(v^{-1}v')w^{-1} \in w(C(\Lambda, c_j)\Gamma)w^{-1}$ as claimed. So given $j \in I(i)$ conjugations of c_j in Λ produce $[C_i : w(C(\Lambda, c_j)\Gamma)w^{-1}]$ pairwise nonconjugate reflections in Γ_σ. Thus, as $wC_jw^{-1} = C_i$ we see that given $j \in I(i)$ conjugations of c_j in Λ produce

$$[C_j : C(\Lambda, c_j)\Gamma] = [C_j/\Gamma : C(\Lambda, c_j)\Gamma/\Gamma] = [C(\theta(\Lambda), \theta(c_j)) : \theta(C(\Lambda, c_j))]$$

pairwise nonconjugate reflections in Γ_σ.

Now in order to complete the proof it suffices to observe that given distinct $j, j' \in I(i)$, the reflections in Γ_σ produced in the above way by reflections c_j and $c_{j'}$ of Λ are nonconjugate in Γ_σ as c_j and $c_{j'}$ are nonconjugate in Λ. □

Theorem 3.2. *Let X be a Riemann surface represented as $\mathscr{H}/\Gamma$ and let $G = \mathrm{Aut}^\pm(X) = \Lambda/\Gamma$ be the group of its automorphisms. Then*

$$\|X\| = \sum_{j \in I_\sim} [\theta(\Lambda) : \theta(C(\Lambda, c_j))].$$

Proof. Each oval of X corresponds to a reflection of Λ and so to c_j^w for some $j \in I$. Observe that if $j \sim j'$ then for arbitrary $w \in \Lambda, c_{j'}^w = c_j^v$ for some $v \in \Lambda$. Thus, we have to count down the ovals of the form c_j^w, where j runs over the set $I_\sim$ of representatives of $I/_\sim$.

Now given $v, w \in \Lambda$ we can show as in the proof of the previous theorem that c_j^w and c_j^v give the same oval if and only if $v^{-1}w \in C(\Lambda, c_j)\Gamma$. Consequently conjugates of c_j give rise to exactly

$$[\Lambda : C(\Lambda, c_j)\Gamma] = [\Lambda/\Gamma : C(\Lambda, c_j)\Gamma/\Gamma] = [\theta(\Lambda) : \theta(C(\Lambda, c_j))]$$

distinct ovals. Finally, as distinct $j, j' \in I_\sim$ produce in the above way distinct ovals, we are finished. □

Corollary 3.3. *Let X be a Riemann surface and let $G = \mathrm{Aut}^{\pm}(X)$. Then $\|X\| \leq |G| \cdot |I_\sim|/2$.*

Let $\widetilde{T}$ be the set of the conjugacy classes of symmetries of X. Actually, Natanzon looked for another invariant of X, namely $\sharp X \sharp = \sum_{\sigma \in \widetilde{T}} \|\sigma\|$. However, the only estimation he was able to find for $\sharp X \sharp$ is that given by $\|X\|$ and the obvious relation $\sharp X \sharp \leq \|X\|$. As a consequence of Theorem 3.1, we obtain at once the formula for $\sharp X \sharp$ which can be a good starting point for obtaining a better estimate in a direct way.

Theorem 3.4. *Let X be a Riemann surface represented as $\mathscr{H}/\Gamma$ and let $G = \mathrm{Aut}^{\pm}(X) = \Lambda/\Gamma$ be the group of its automorphisms. Then*

$$\sharp X \sharp = \sum_{j \in I_\sim} [C(\theta(\Lambda), \theta(c_j)) : \theta(C(\Lambda, c_j))].$$

Proof. Indeed, $\sharp X \sharp = \sum_{i \in I_\approx} k_i = \sum_{i \in I_\approx} \sum_{j \in I(i)} [C(\theta(\Lambda), \theta(c_j)) : \theta(C(\Lambda, c_j))] = \sum_{j \in I_\sim} [C(\theta(\Lambda), \theta(c_j)) : \theta(C(\Lambda, c_j))]$. □

Though the above theorems have fairly technical character they are very useful in concrete applications. A prototype of Theorem 3.1 applied to the case of the triangle *NEC* group $(0; +; [-]; \{(2, 3, 7)\})$ was used in [4] to study topological types of symmetries of Riemann surfaces on which $\mathrm{PSL}(2, q)$ acts as a Hurwitz group of automorphisms and in [8], we used Theorem 3.1 to study a problem concerning types of symmetries of Riemann surfaces with large groups of automorphisms. In the next section we shall deal with the applications of Theorem 3.2.

From the above results we see that all we need to find an estimate for $\|X\|$ is to know $\mathrm{Aut}^{\pm}(X) = \Lambda/\Gamma$ and the centralizers of the reflections in Λ. Singerman [23] proved that the centralizer of a reflection c of an *NEC* group is infinite and claimed that it is isomorphic to $Z_2 \oplus Z$ if c corresponds to an empty period cycle or to a period cycle in which all periods are odd and to $Z_2 \oplus (Z_2 * Z_2)$ otherwise. Going a bit more into the details of the Singerman's proof (cf. [21]), one can find explicitly the generators for these groups. We shall need them in the next section for *NEC* groups with special signatures in order to find a more precise, as used in Corollary 3.3, estimation for $|\theta(C(\Lambda, c_i))|$. However, in most cases the following lemma will be sufficient.

Lemma 3.5. *Let c_i be a canonical reflection of Λ corresponding to an even link period. Then $|\theta(C(\Lambda, c_i))| \geq 4$.*

Proof. Let c_{i-1} be another canonical reflection of Λ such that $c_{i-1}c_i$ has even order n. Then it is easy to check that both c_i and $(c_{i-1}c_i)^{n/2}$ are in $C(\Lambda, c_i)$. Moreover, $\theta(c_i) \neq 1$ and $\theta(c_{i-1}c_i)^{n/2} \neq 1$ since otherwise Γ would have an element of finite order. Furthermore, $\theta(c_i) \neq \theta(c_{i-1}c_i)^{n/2}$ since in the other case $c_i(c_{i-1}c_i)^{n/2}$ would be an orientation-reversing element of Γ. □

Corollary 3.6. *With the notations adopted at the beginning of this section, let Λ be an NEC group without empty period cycles each with some even link period. Then $|I_\sim|$ is the number of even link periods and $\|X\| \leq |G| \cdot |I_\sim|/4$.*

Proof. The first part follows from the fact that two canonical reflections corresponding to an odd link period are conjugate. So $I_\sim$ can be so chosen that for $i \in I_\sim$, c_i corresponds to an even link period and so $|\theta(C(\Lambda, c_i))| \geq 4$ by the previous lemma. Thus, the result follows by Theorem 3.2. □

4. Applications

In this section we shall use Theorem 3.2 to find an upper estimate for the function ν defined in the Section 1. Throughout the section $X = \mathscr{H}/\Gamma$ and $G = \mathrm{Aut}^{\pm}(X) = \Lambda/\Gamma$. Assume that Λ has a signature of the general form

$$(g'; \pm; [m_1, \ldots, m_r]; \{(-), \overset{k}{\ldots}, (-), (n_{11}, \ldots, n_{1s_1}), \ldots, (n_{m1}, \ldots, n_{ms_m})\}). \tag{8}$$

Then

$$\mu(\Lambda) \geq 2\pi \left(\alpha g' + 3k/4 + m - 2 + r/2 + \frac{1}{4}\left(k + \sum_{i=1}^{m} s_i\right)\right)$$

and therefore

$$|G| \leq \left(\frac{2(g-1)}{\alpha g' + 3k/4 + m - 2 + r/2 + \frac{1}{4}\left(k + \sum_{i=1}^{m} s_i\right)}\right). \tag{9}$$

So by Corollary 3.3,

$$\begin{aligned} \|X\| &\leq \frac{|G|}{2}\left(k + \sum_{i=1}^{m} s_i\right) \\ &\leq (g-1)\left(\frac{\left(k + \sum_{i=1}^{m} s_i\right)}{(\alpha g' + 3k/4 + m - 2 + r/2) + \frac{1}{4}\left(k + \sum_{i=1}^{m} s_i\right)}\right). \end{aligned} \tag{10}$$

Theorem 4.1. *Let X be a Riemann surface of genus $g \geq 2$, $g \neq 2, 3, 5, 7, 9$. Then $\|X\| \leq 12(g-1)$. Furthermore, the maximum number of ovals that a Riemann surface of genus $g = 2, 3, 5, 7$ and 9 admits is $24, 36, 72, 126$ and 100, respectively.*

Proof. We must study the case $\alpha g' + 3k/4 + m + r/2 < 2$ since otherwise $\|X\| \leq 4(g-1)$ by (10). But if $\alpha g' = 1$ then $m = 0$ and as $k \geq 1$ we get $r = 0, k = 1$, i.e, $\mu(\Lambda) = 0$. Thus $g' = 0$ and so $3k + 4m + 2r < 8$. If $m = 0$ and $k = 2$ then $r = 0$ i.e, $\mu(\Lambda) = 0$. If $m = 0$, $k = 1$ and $r \leq 1$ then $\mu(\Lambda) < 0$ whilst if $r = 2$ then then $m_1 > 2$ or $m_2 > 2$, since in the other case $\mu(\Lambda) = 0$. However in such a case $\mu(\Lambda) \geq \pi/3$ and thus $\|X\| \leq |G|/2 \leq 6(g-1)$. If $m = 1$ and $k = 1$, then $\mu(\Lambda) \geq 2\pi s_1/4$. So $|G| \leq 8(g-1)/s_1$ and therefore $\|X\| \leq (4(s_1+1)/s_1)(g-1) \leq 8(g-1)$. So assume that $m = 1$ and $k = 0$, i.e, let Λ be an *NEC* group with signature

$$(0; +; [m_1, \ldots, m_r]; \{(n_1, \ldots, n_s)\}).$$

We have $r \leq 1$.

First let $s = 1$. Then $r = 1$ since otherwise $\mu(\Lambda) < 0$ and so Λ is an *NEC* group with signature

$$(0; +; [m]; \{(n)\}).$$

Clearly $m \geq 3$. If $n = 2$, then $m \geq 5$ and thus $\mu(\Lambda) \geq \pi/10$. But then $\|X\| \leq |G|/4 \leq 10(g-1)$ by Corollary 3.6. If $n = 3$, then $m \geq 4$ and so $\mu(\Lambda) \geq \pi/6$. Thus $|G| \leq 24(g-1)$ and therefore $\|X\| \leq 12(g-1)$ by Corollary 3.3. If $n = 4$, then $m \geq 3$. So $\mu(\Lambda) \geq \pi/12$ and therefore $\|X\| \leq |G|/4 \leq 12(g-1)$. Now let $n = 5$. Then for $m \geq 4$, $\|X\| \leq |G|/2 \leq (20/3)(g-1)$. For $n = 5$ and $m = 3$, $|G| = 30(g-1)$. However, $c_0, e(c_0c_1)^2 \in C(\Lambda, c_0)$ and $\theta(e(c_0c_1)^2) \neq \theta(c_0)$. Furthermore $\theta(e(c_0c_1)^2) \neq 1$, as $\theta(e)$ has order 3 while $\theta((c_0c_1)^2)$ has order 5. So $|\theta(C(\Lambda, c_0))| \geq 4$ and therefore $\|X\| \leq |G|/4 < 8(g-1)$. For $n = 6$, $\|X\| \leq |G|/4 \leq 6(g-1)$. Finally, for $n \geq 7$, $\|X\| \leq |G|/2 < 11(g-1)$.

Now let $s = 2$. Then $r = 1$, since otherwise $\mu(\Lambda) < 0$ and so Λ have signature

$$(0; +; [m]; \{(n_1, n_2)\}).$$

If $m \geq 3$, then $\|X\| \leq 2|G|/2 \leq 12(g-1)$. Assume then that $m = 2$. If n_1, n_2 are odd, $\mu(\Lambda) \geq \pi/3$ and thus $\|X\| \leq 6(g-1)$ as $|G| \leq 12(g-1)$. If n_1 is even and n_2 is odd, then $\mu(\Lambda) \geq \pi/6$. So $\|X\| \leq |G|/4 \leq 6(g-1)$. Finally, if both of n_1, n_2 are even, then one of them is at least 4 since otherwise $\mu(\Lambda) = 0$. But in such a case $\mu(\Lambda) \geq \pi/4$ and therefore $\|X\| \leq 2|G|/4 \leq 8(g-1)$.

Assume now that $s \geq 4$. Then for $r > 0$, $\mu(\Lambda) \geq \pi(s-2)/2$. Thus, $|G| \leq 8(g-1)/(s-2)$ and therefore $\|X\| \leq s|G|/2 \leq 4s(g-1)/(s-2) \leq 8(g-1)$. Hence, we can assume $r = 0$, i.e., that Λ has signature

$$(0; +; [-]; \{(n_1, \ldots, n_s)\})$$

which we shall abbreviate $(n_1, \ldots, n_s)$.

If all n_i are odd, then $\mu(\Lambda) \geq 2\pi/3$. Consequently, $|G| \leq 6(g-1)$ and $\|X\| \leq 3(g-1)$. Thus, assume that some n_i are even and observe that as $\mu(\Lambda) \geq \pi(s-4)/2$, $|G| \leq 8(g-1)/(s-4)$ and therefore for $s \geq 5$ $\|X\| \leq (2s/(s-4))(g-1) \leq 10(g-1)$ by Corollary 3.6.

So let $s = 4$ and let c_0, c_1, c_2, c_3 be the set of canonical reflections for Λ. If at least two of the link periods are odd, then $\mu(\Lambda) \geq \pi/3$. Therefore, $|G| \leq 12(g-1)$

and thus $\|X\| \leq 6(g-1)$ by Corollary 3.3. So we can assume that at least three of the link periods, say n_1, n_2, n_3, are even. Now if n_4 is odd and $n_i \geq 4$ for some $i \leq 3$ then $\mu(\Lambda) \geq 5\pi/12$. So $|G| \leq (48/5)(g-1)$ and therefore $\|X\| \leq 3|G|/4 < 8(g-1)$. If n_4 is even and at least two of the link periods are ≥ 4, $\mu(\Lambda) \geq \pi/2$ and thus $\|X\| \leq 4|G|/4 \leq 8(g-1)$ by Corollary 3.6. So, for $s=4$ it remains to consider the case of Λ with signature $(2,2,2,n)$.

Clearly, $n \geq 3$ and $|G| = (8n/(n-2))(g-1)$ as $\mu(\Lambda) = \pi(n-2)/(2n)$. Now $c_0, c_1, c_2 \in C(\Lambda, c_1)$, $\theta(c_0) \neq \theta(c_1)$, $\theta(c_2) \neq \theta(c_1)$ and $\theta(c_0)\theta(c_2) \neq \theta(c_1)$. Moreover, $\theta(c_0) \neq \theta(c_2)$ since otherwise $(c_0c_3)^2 \in \Gamma$ would be an element of finite order. Thus, $|\theta(C(\Lambda, c_1))| \geq 8$ and in the same way one can prove that $|\theta(C(\Lambda, c_2))| \geq 8$. So $\|X\| \leq 2|G|/8 + 2|G|/4 \leq (6n/(n-2))(g-1) \leq 12(g-1)$ for $n \geq 4$. Also for $n=3$, $\|X\| \leq 2|G|/8 + |G|/4 = (4n/(n-2))(g-1) = 12(g-1)$.

Finally let $s=3$. If $r>0$, then $\mu(\Lambda) \geq \pi/2$ and therefore $\|X\| \leq 3|G|/2 \leq 12(g-1)$ by Corollary 3.3. So we have proved that $\|X\| \leq 12(g-1)$ except for Λ with signature

$$(k, l, m).$$

By [11] we can assume, without loss of generality, that $k \leq l \leq m$. Let c_0, c_1, c_2 be a system of canonical reflections for Λ.

If k, l and m are odd, then $\|X\| \leq |G|/2 \leq 11(g-1)$ or else Λ has signature $(3,3,5)$. In the last case $|G| = 30(g-1)$. However, it is easy to check that $c_0, (c_2c_0)^2(c_1c_2)(c_0c_1) \in C(\Lambda, c_0)$. Furthermore, $\theta(c_0) \neq \theta((c_2c_0)^2(c_1c_2)(c_0c_1))$, since in the other case we would have an orientation-reversing element in Γ and $\theta((c_2c_0)^2$ $(c_1c_2)(c_0c_1)) \neq 1$ since in such a case one can show, using the defining relations for Λ, that $\theta(c_0c_2) = 1$ which is also imposible. So $|\theta(C(\Lambda, c_0))| \geq 4$ and therefore $\|X\| \leq (15/2)(g-1)$, as all reflections of Λ are conjugated. Assume then that some of k, l and m are even. As before we can show, using Corollary 3.6, that $\|X\| \leq 12(g-1)$ for $k \geq 3$.

Thus, assume $k=2$ and recall that we have assumed $l \leq m$. Observe first that for $l \geq 8$, $\mu(\Lambda) \geq \pi/4$ and so $\|X\| \leq 3|G|/4 = 12(g-1)$. If $l=7$, then $\mu(\Lambda) \geq 3\pi/14$. So $|G| \leq (56/3)(g-1)$ and $\|X\| \leq 2|G|/4 < 10(g-1)$.

Let $l=6$. Then $\mu(\Lambda) = (m-3)\pi/(3m)$ and thus $|G| = (12m/(m-3))(g-1)$. Therefore for odd m, $\|X\| \leq (6m/(m-3))(g-1) < 11(g-1)$. Assume thus that m is even. Then $\|X\| \leq 3|G|/4$ and so $\|X\| \leq 12(g-1)$ for $m \geq 12$. Thus, let $m = 2m'$ for $3 \leq m' \leq 5$. Then $(c_0c_1), (c_1c_2)^3 \in C(\Lambda, c_1)$ and their images under θ are distinct, since otherwise $\theta(c_0c_2) = \theta(c_1c_2)^4$ which is impossible, as the first has order m while the other has order 3. Furthermore, $\theta(c_1) \notin \langle \theta(c_0c_1), \theta(c_1c_2)^3 \rangle$. So $|\theta(C(\Lambda, c_1))| \geq 8$. Similarly, one can show that $|\theta(C(\Lambda, c_0))| \geq 8$. Thus, $\|X\| \leq 2|G|/8 + |G|/4 = (6m/(m-3))(g-1) \leq 12(g-1)$.

For $l=5$, $\mu(\Lambda) = (3m-10)\pi/(10m)$ and thus $|G| = (40m/(3m-10))(g-1)$. If m is odd, then $\|X\| \leq |G|/4 \leq 10(g-1)$. If m is even, $\|X\| \leq 2|G|/4 < 12(g-1)$ for $m \geq 8$ and so we have to consider the case $m=6$. Here $|G| = 30(g-1)$. Now $(c_0c_1), (c_0c_2)^3 \in C(\Lambda, c_0)$ and again their images under θ are distinct since otherwise $\theta(c_2c_1) = \theta(c_0c_2)^2$ which is imposible, as the first has order 5 while the other has order 3. As before we deduce that $|\theta(C(\Lambda, c_0))| \geq 8$ and therefore $\|X\| \leq |G|/8 + |G|/4 < 12(g-1)$. So it remains to deal only with the cases $l = 3$ and $l = 4$.

Here we shall need an exact description of the centralizer of the reflections in Λ. As we mentioned in the last part of the previous section it is possible to find generators for the centralizers of the reflections. According to the parity of k, l and m we have (cf. [21, 23]):
for $k = 2k'$, $l = 2l'$, $m = 2m'$,

$$\begin{aligned} C(\Lambda, c_0) &= \langle c_0 \rangle \oplus (\langle (c_0c_1)^{k'} \rangle * \langle (c_0c_2)^{m'} \rangle), \\ C(\Lambda, c_1) &= \langle c_1 \rangle \oplus (\langle (c_0c_1)^{k'} \rangle * \langle (c_1c_2)^{l'} \rangle), \\ C(\Lambda, c_2) &= \langle c_2 \rangle \oplus (\langle (c_0c_2)^{m'} \rangle * \langle (c_1c_2)^{l'} \rangle) \end{aligned} \tag{11}$$

for $k = 2k'$, $l = 2l'$, $m = 2m' + 1$,

$$\begin{aligned} C(\Lambda, c_0) &= \langle c_0 \rangle \oplus (\langle (c_0c_1)^{k'} \rangle * \langle (c_2c_0)^{m'}(c_1c_2)^{l'}(c_0c_2)^{m'} \rangle) \\ C(\Lambda, c_1) &= \langle c_1 \rangle \oplus (\langle (c_0c_1)^{k'} \rangle * \langle (c_1c_2)^{l'} \rangle) \end{aligned} \tag{12}$$

for $k = 2k'$, $l = 2l' + 1$, $m = 2m' + 1$,

$$C(\Lambda, c_0) = \langle c_0 \rangle \oplus (\langle (c_0c_1)^{k'} \rangle * \langle (c_2c_0)^{m'}(c_1c_2)^{l'}(c_0c_1)^{k'}(c_2c_1)^{l'}(c_0c_2)^{m'} \rangle) \tag{13}$$

for $k = 2k'$, $l = 2l' + 1$, $m = 2m'$,

$$\begin{aligned} C(\Lambda, c_0) &= \langle c_0 \rangle \oplus (\langle (c_0c_1)^{k'} \rangle * \langle (c_0c_2)^{m'} \rangle), \\ C(\Lambda, c_1) &= \langle c_1 \rangle \oplus (\langle c_0c_1)^{k'} \rangle * \langle (c_2c_1)^{l'}(c_0c_2)^{m'}(c_1c_2)^{l'} \rangle), \end{aligned} \tag{14}$$

where $*$ denotes the operation of the free product.

We shall see below that the group G of all automorphisms of a Riemann surface which has more than $12(g-1)$ ovals is a factor group of some group $\widetilde{G}$ with a certain specific presentation which makes it finite. In virtue of the Hurwitz Riemann formula, the order of $\widetilde{G}$ gives restrictions on the genera of the corresponding Riemann surfaces. In most cases the calculation of the order was rather an easy matter. In one particular case however, the use of computer was necessary. We acknowledge that all the results obtained have been checked by the GAP Programm (Groups, Algorithms and Programming) developed by J. Neubüser's group at Aachen.

For $l = 4$, $\mu(\Lambda) = (m-4)\pi/(4m)$ and thus $|G| = (16m/(m-4))(g-1)$. Now if $|\theta(C(\Lambda, c_1))| < 8$, then by (11) or (12), $\theta(c_0c_1) = \theta(c_1c_2)^2$ and therefore $\theta(c_0c_2) = \theta(c_1c_2)^3$. Thus, $\theta(c_0c_2)^4 = 1$, which implies that m divides 4, an absurd since in such a case $\mu(\Lambda) \leq 0$. So $|\theta(C(\Lambda, c_1))| \geq 8$. Now if m is even, then similar arguments show that $|\theta(C(\Lambda, c_0))| \geq 8$. So $\|X\| \leq 2|G|/8 + |G|/4 = (8m/(m-4))(g-1) \leq 12(g-1)$ for $m \geq 12$. For $m = 10, |\theta(C(\Lambda, c_2))| \geq 8$, since otherwise $\theta(c_0c_2)^5 = \theta(c_1c_2)^2$ and thus $\theta(c_0c_2)^4 = \theta(c_1)\theta(c_2c_0)\theta(c_1)$, which implies $\theta(c_0c_2)^5 = 1$, an absurd. Similarly, $|\theta(C(\Lambda, c_2))| \geq 8$ for $m = 6, 8$. So $\|X\| \leq 3|G|/8 \leq 12(g-1)$ for $m = 8, 10$. Thus, we have to deal with the case $m = 6$. Observe that $|G| = 48(g-1)$ and $\|X\| \leq 18(g-1)$, as we already know that $|\theta(C(\Lambda, c_i))| \geq 8$ for $i = 0, 1, 2$. If $|\theta(C(\Lambda, c_i)| > 8$ for

$i=0,1,2$, then $|\theta(C(\Lambda,c_i)| \geq 12$ and $\|X\| \leq 3|G|/12 = 12(g-1)$. Assume then that $|\theta(C(\Lambda,c_i))| = 8$ for some $i=0,1,2$.

If $|\theta(C(\Lambda,c_0))| = 8$, then by (11), $\theta(c_0c_1)\theta(c_0c_2)^3$ is an element of order 2 and thus G is a factor group of the group $\widetilde{G}$ with presentation

$$\langle c_0,c_1,c_2 \mid c_0^2, c_1^2, c_2^2, (c_0c_1)^2, (c_1c_2)^4, (c_0c_2)^6, ((c_0c_1)(c_0c_2)^3)^2\rangle,$$

which can be easily shown to have order 96. So $g=2$ or $g=3$. However, later on we shall show that there exist Riemann surfaces of genera 2 and 3, with 24 and 36 ovals, respectively.

Now if $|\theta(C(\Lambda,c_1))| = 8$, then $\theta(c_0c_1)\theta(c_1c_2)^2$ is an element of order 2 and thus G is a factor group of the group with presentation

$$\langle c_0,c_1,c_2 \mid c_0^2, c_1^2, c_2^2, (c_0c_1)^2, (c_1c_2)^4, (c_0c_2)^6((c_0c_1)(c_1c_2)^2)^2\rangle,$$

which again can be shown to have order 48 and so here $g=2$.

Finally, assume that $|\theta(C(\Lambda,c_i))| > 8$ for $i=0,1$ and let $|\theta(C(\Lambda,c_2))| = 8$. Then $\|X\| \leq 2|G|/12 + |G|/8 = 14(g-1)$. As before we argue that G is a factor group of the group $\widetilde{G}$ with presentation

$$\langle c_0,c_1,c_2 \mid c_0^2, c_1^2, c_2^2, (c_0c_1)^2, (c_1c_2)^4, (c_0c_2)^6, ((c_1c_2)^2(c_0c_2)^3)^2\rangle,$$

which can be shown to have order 192. So here g can be only 2,3 or 5 and later on we shall see that there exists a Riemann surface of genus 5 with 72 ovals.

So let m be odd. Then $\|X\| \leq |G|/8 + |G|/4 < 11(g-1)$ for $m \geq 9$ and therefore we have to consider here only the cases $m=5$ and $m=7$. Assume first that $m=7$. Then $|\theta(C(\Lambda,c_0))| \geq 8$. Indeed, if this is not the case, then by (12), $\theta(c_0c_1) = \theta(c_2c_0)^3\theta(c_1c_2)^2\theta(c_0c_2)^3 = \theta(c_0c_2)^3\theta(c_1)\theta(c_0c_2)\theta(c_1)\theta(c_0c_2)^3$. Therefore, $\theta(c_1c_2)$ and $\theta(c_0c_2)$ are conjugate which is impossible as they have distinct orders. So $\|X\| \leq 2|G|/8 \leq (4m/(m-4))(g-1) < 10(g-1)$. Finally, assume $m=5$. Then similarly as above we can show that $|\theta(C(\Lambda,c_0))| \geq 8$. We already know that $|\theta(C(\Lambda,c_1))| \geq 8$ and we shall show that $|\theta(C(\Lambda,c_1))| \geq 16$. Indeed, if this is not the case, then $\theta(c_0c_1)$ $\theta(c_1c_2)^2$ has order 2 or 3. However, one can easily show that in the first case $\theta(c_1c_2)^2 = 1$ while in the other one $\theta(c_1c_2) = 1$ which are impossible. Now if $|\theta(C(\Lambda,c_1))| = 16$, then $\|X\| \leq |G|/16 + |G|/8 = 15(g-1)$ and G is a factor group of the group with presentation

$$\langle c_0,c_1,c_2 \mid c_0^2, c_1^2, c_2^2, (c_0c_1)^2, (c_1c_2)^4, (c_0c_2)^5, ((c_0c_1)(c_1c_2)^2)^4\rangle,$$

which can be shown to have order 320. So here g can be equal 2,3 or 5 only. Thus, assume that $|\theta(C(\Lambda,c_1))| > 16$. Then $|\theta(C(\Lambda,c_1))| \geq 20$ and we shall look now for the order of $\theta(C(\Lambda,c_0))$. If $|\theta(C(\Lambda,c_0))| = 8$ then $\|X\| \leq |G|/20 + |G|/8 = 14(g-1)$ and G is a factor group of the group $\widetilde{G}$ with presentation

$$\langle c_0,c_1,c_2 \mid c_0^2, c_1^2, c_2^2, (c_0c_1)^2, (c_1c_2)^4, (c_0c_2)^5, ((c_0c_1)(c_2c_0)^2(c_2c_1)^2(c_0c_2)^2)^2\rangle,$$

which can be checked to have order 640. So here $g = 2,3,5$ or 9 only. For $g=2,3,5$ there exist Riemann surfaces with more than $14(g-1)$ ovals. So let $g=9$. Then $G=\widetilde{G}$.

However, $(c_0c_1)(c_1c_2)^2$ represents in $\widetilde{G}$ an element of order 8. So $|\theta(C(\Lambda,c_1))| = 32$, by (12), and thus for $g = 9$ there exists a Riemann surface having $|G|/8 + |G|/32 = (25/2)(g-1) = 100$ ovals. Finally, if $|\theta(C(\Lambda,c_0))| > 8$, then $|\theta(C(\Lambda,c_0))| \geq 12$ and so $\|X\| \leq |G|/12 + |G|/20 < 11(g-1)$.

Thus, consider the case of Λ with signature

$$(2,3,m).$$

Here $\mu(\Lambda) = (m-6)\pi/(6m)$ and so $|G| = (24m/(m-6))(g-1)$.

Let first $m = 2m'$ for some $m' \geq 4$. We shall show that $|\theta(C(\Lambda,c_i))| \geq 8$ for $i = 0,1$. Indeed, if $|\theta(C(\Lambda,c_0))| < 8$, then by (14) $\theta(c_0c_1) = \theta(c_0c_2)^4$ which implies $\theta(c_2c_1) = \theta(c_0c_2)^3$ and so $\theta(c_0c_2)^9 = 1$. Now if $|\theta(C(\Lambda,c_1))| < 8$, then again by (14), $\theta(c_0c_1) = \theta(c_2c_1)\theta(c_2c_0)^{m'}\theta(c_1c_2)$. Now using the defining relations for Λ we can easily show that $\theta(c_0c_2)^2 = \theta(c_1)\theta(c_0c_2)^{m'+1}\theta(c_1)$. However, the last implies $\theta(c_0c_2)^4 = \theta(c_1)\theta(c_0c_2)^2\theta(c_1)$ and in consequence $\theta(c_0c_2)^6 = 1$, which is impossible. Thus $\|X\| \leq 2|G|/8 \leq 12(g-1)$ for $m \geq 12$. So in case m is even it remains to deal with the cases $m = 8$ and $m = 10$. Assume first that $m = 8$. We know that $|\theta(C(\Lambda,c_i))| \geq 8$ for $i = 0,1$. Thus, $\|X\| \leq 2|G|/8 = 24(g-1)$. If $|\theta(C(\Lambda,c_0))| = 8$, then G is a factor group of the group $\widetilde{G}$ with presentation

$$\langle c_0,c_1,c_2 \mid c_0^2,c_1^2,c_2^2,(c_0c_1)^2,(c_1c_2)^3,(c_0c_2)^8,((c_0c_1)(c_0c_2)^4)^2\rangle.$$

So $G = \widetilde{G}$ as the last has order 96. Thus, here X has genus 2. Furthermore, $(c_0c_1)(c_2c_1)(c_0c_2)^4(c_1c_2)$ represents in G an element of order 2 and thus in virtue of (14), $|\theta(C(\Lambda,c_1))| = 8$. So there exists a Riemann surface of genus $g = 2$ with $2|G|/8 = 24$ ovals. Thus, assume that $|\theta(C(\Lambda,c_0))| > 8$. Then $|\theta(C(\Lambda,c_0))| > 12$, since otherwise $(\theta(c_0c_1)\theta(c_0c_2)^4)^3 = 1$ while using the defining relations for Λ we can show that the last implies $\theta(c_1c_2) = 1$, which is impossible. So $|\theta(C(\Lambda,c_0))| \geq 16$.

Now if $|\theta(C(\Lambda,c_1))| = 8$, then G is a factor group of the group $\widetilde{G}$ with the presentation

$$\langle c_0,c_1,c_2 \mid c_0^2,c_1^2,c_2^2,(c_0c_1)^2,(c_1c_2)^3,(c_0c_2)^8,((c_0c_1)(c_2c_1)(c_0c_2)^4(c_1c_2))^2\rangle,$$

which can be shown to have order 384. Thus, here X has genus $g = 2,3$ or 5 as $|G| = 96(g-1)$. We have already shown that there exists a Riemann surface of genus $g = 2$ with 24 ovals. If $g = 5$, then $G = \widetilde{G}$, while $(c_0c_1)(c_0c_2)^4$ represents in $\widetilde{G}$ an element of order 4 and thus in virtue of (14), $|\theta(C(\Lambda,c_0))| = 16$. So there exsists a Riemann surface of genus $g = 5$ with $|G|/8 + |G|/16 = 18(g-1) = 72$ ovals. Finally, we shall show that there exists a Riemann surface of genus $g = 3$ with $18(g-1)$ ovals. Indeed, one can show that the group G with presentation

$$\langle c_0,c_1,c_2 | c_0^2,c_1^2,c_2^2,(c_1c_2)^3,(c_0c_2)^8,[((c_0c_1)(c_1c_2))^2,((c_1c_2)(c_0c_1))^2]\rangle$$

has order 192. Moreover, $(c_0c_1)(c_0c_2)^4$ and $(c_0c_1)(c_2c_1)(c_0c_2)^4(c_1c_2)$ represent in G elements of order 4 and 2, respectively, and thus in virtue of (14), $|\theta(C(\Lambda,c_0))| = 16$ and $|\theta(C(\Lambda,c_1))| = 8$. So the corresponding surface has $|G|/16 + |G|/8 = 18(g-1) = 36$ ovals.

So we can assume that $|\theta(C(\Lambda, c_1))| \geq 12$. Now if $|\theta(C(\Lambda, c_1))| \geq 16$, then $\|X\| \leq 2|G|/16 = 12(g-1)$. So let $|\theta(C(\Lambda, c_1))| = 12$. Then $\|X\| \leq |G|/16+|G|/12 = 14(g-1)$ and G is a factor group of the group $\widetilde{G}$ with presentation

$$\langle c_0, c_1, c_2 \mid c_0^2, c_1^2, c_2^2, (c_0c_1)^2, (c_1c_2)^3, (c_0c_2)^8, ((c_0c_1)(c_2c_1)(c_0c_2)^4(c_1c_2))^3 \rangle$$

which was checked, using GAP, to have order 4896. But using the GAP once more we see that $(c_0c_1)(c_0c_2)^4$ represents in G an element of order 9. So $|\theta(C(\Lambda, c_0))| = 36$ and thus $\|X\| = |G|/12 + |G|/36 < 11(g-1)$. Now we see that the only possible quotients of $\widetilde{G}$ which may be smooth factors of Λ may have orders 1632, 288 and 96. However in a quotient G of order 288 the element $(c_0c_1)(c_0c_2)^4$ would still represent an element of order 9 and therefore also in this case we would have $\|X\| < 11(g-1)$. In quotients of order 96 and 1632, $(c_0c_1)(c_0c_2)^4$ would represent an element of order 3 which we showed to be impossible.

Now assume that $m = 10$. Here $|G| = 60(g-1)$ and as $|\theta(C(\Lambda, c_i))| \geq 8$ for $i = 0, 1$, $\|X\| \leq 2|G|/8 = 15(g-1)$. Thus, we can assume that $g \neq 2, 5$. If $|\theta(C(\Lambda, c_i))| > 8$ for $i = 0, 1$, then $|\theta(C(\Lambda, c_i))| \geq 12$ and so $\|X\| \leq 2|G|/12 = 10(g-1)$. So let $|\theta(C(\Lambda, c_0))| = 8$. Then G is a factor group of the group $\widetilde{G}$ with presentation

$$\langle c_0, c_1, c_2 \mid c_0^2, c_1^2, c_2^2, (c_0c_1)^2, (c_1c_2)^3, (c_0c_2)^{10}, ((c_0c_1)(c_0c_2)^5)^2 \rangle.$$

Here $\widetilde{G}$ has order 240. We have assumed that $g \neq 2, 5$ and we shall see that $g \neq 3$. Indeed, if $g = 3$, then $|G| = 120$. However, by a theorem of Singerman [22], c_0c_1 and c_1c_2 represent in G elements generating a subgroup G' of order 60. Clearly c_0c_1 and c_1c_2 represent in G' elements of orders 2 and 3, respectively, and their product c_0c_2 an element of order 10. But, then it is easy to show that G' contains a normal subgroup of order 2, which obviously is impossible. If $|\theta(C(\Lambda, c_1))| = 8$, then G is a factor group of the group $\widetilde{G}$ with presentation

$$\langle c_0, c_1, c_2 \mid c_0^2, c_1^2, c_2^2, (c_0c_1)^2, (c_1c_2)^3, (c_0c_2)^{10}, ((c_0c_1)(c_2c_1)(c_0c_2)^5(c_1c_2))^2 \rangle,$$

which has order 720. So $g = 2, 3, 4, 5, 7$ or 13. We already know that there exist Riemann surfaces of genera $g = 2, 3, 5$ with more than $15(g-1)$ ovals. Later on we shall show that there exists a Riemann surface of genus $g = 7$ with 126 ovals. So we have to deal with the cases $g = 4$ and $g = 13$ only. In the later case $G = \widetilde{G}$. However, $(c_0c_1)(c_0c_2)^5$ represents in $\widetilde{G}$ an element of order 6. So $|\theta(C(\Lambda, c_0))| = 24$ by (14) and thus $\|X\| = 10(g-1)$. If $g = 4$, then $|G| = 120$ and as in the case $|\theta(C(\Lambda, c_0))| = 8$ we can show that this is impossible.

Finally, let $m = 2m'+1$. Then $|\theta(C(\Lambda, c_0))| \geq 8$, since otherwise by (13), $\theta(c_0c_1) = \theta(c_0c_2)^{m'+1}\theta(c_1c_2)\theta(c_1c_0)\theta(c_2c_1)\theta(c_0c_2)^{m'}$. But, then using the defining relations for Λ one can show as before that $\theta(c_1c_2)$ and $\theta(c_0c_2)^2$ are conjugate, which implies $\theta(c_0c_2)^6 = 1$, an absurd. Hence, $|\theta(C(\Lambda, c_0))| \geq 8$ indeed and therefore $\|X\| \leq |G|/8 = (3m/(m-6))(g-1) \leq 9(g-1)$ for $m \geq 9$. So it remains to consider the case $m = 7$. Here $\|X\| \leq 21(g-1)$ and the bound is attained if and only if $|\theta(C(\Lambda, c_0))| = 8$ while the last is equivalent to $\theta((c_0c_1)(c_2c_0)^3(c_1c_2)(c_1c_0)(c_2c_1)(c_0c_2)^3)^2 = 1$ and therefore to the fact that G is a quotient group of the group $\widetilde{G}$ with presentation

$$\langle c_0, c_1, c_2 \mid c_0^2, c_1^2, c_2^2, (c_0c_1)^2, (c_1c_2)^3, (c_0c_2)^7,$$
$$((c_0c_1)(c_2c_0)^3(c_1c_2)(c_1c_0)(c_2c_1)(c_0c_2)^3)^2\rangle,$$

which can be shown to have order 1008. Now by a theorem of Singerman [22], c_0c_1 and c_1c_2 generate a subgroup of $\widetilde{G}$ order 504 which is isomorphic to PSL(2,8) as a Hurwitz group. So in particular $G = \widetilde{G}$. This proves that there exists a Riemann surface of genus $g = 7$ with $21(g-1) = 126$ ovals.

Now if $|\theta(C(\Lambda, c_0))| > 8$ then $|\theta(C(\Lambda, c_0))| \geq 12$ and so we have $\|X\| \leq |G|/12 = (2m/(m-6))(g-1) \leq 14(g-1)$. However, the last bound is attained if and only if $\theta((c_0c_1)(c_2c_0)^3(c_1c_2)(c_1c_0)(c_2c_1)(c_0c_2)^3)$ has order 3 and therefore if and only if G is a factor group of the group $\widetilde{G}$ with presentation

$$\langle c_0, c_1, c_2 \mid c_0^2, c_1^2, c_2^2, (c_0c_1)^2, (c_1c_2)^3, (c_0c_2)^7,$$
$$((c_0c_1)(c_2c_0)^3(c_1c_2)(c_1c_0)(c_2c_1)(c_0c_2)^3)^3\rangle,$$

which can be shown to have order 336. As before we argue that $G = \widetilde{G}$ and therefore X has genus 3.

Finally if $|\theta(C(\Lambda, c_0))| > 12$, then $|\theta(C(\Lambda, c_0))| \geq 16$ and so we have $\|X\| \leq |G|/16 < 11(g-1)$. □

In terms of the function v we can state

Theorem 4.2. *Let v be the function defined in the introduction. Then $v(g) \leq 12(g-1)$ for $g \neq 2, 3, 5, 7, 9$. For $g = 2, 3, 5, 7$ and 9, $v(g) = 24, 36, 72, 126$ and 100, respectively. Moreover, $v(g) = 12(g-1)$ for all values of g of the form $g = 8m^2 + 1, m \geq 2$.*

Proof. The first part is an immediate consequence of Theorem 4.1. In order to prove the second part, consider the group $\Omega = Z_2 * Z_3 = \langle x, y | x^2, y^3\rangle$ and let M be the subgroup of Ω generated by the images of $A = [x, y]$ and $B = [x, y^{-1}]$. It is easy to check that

$$\begin{aligned} &A^x = A^{-1}, \quad B^x = B^{-1}, \\ &A^y = A^{-1}B, \quad B^y = A^{-1}. \end{aligned} \tag{15}$$

So M is a normal subgroup of Ω. By the Kurosh subgroup theorem M is free. Now M is noncyclic. Indeed assume on the contrary that $A = C^\alpha$ and $B = C^\beta$ for some $C \in M$ and some integers α and β. Then $A^{-1}B = C^{\beta-\alpha}$. Now $C^y = C$ and so $A = (A^{-1}B)^{y^{-1}} = (C^{y^{-1}})^{\beta-\alpha} = C^{\beta-\alpha} = A^{-1}B$. But then $B = A^2$ and thus $(A^{-1}B)^2 = (A^2)^y = B^y = A^{-1}$. Hence, $A = B^2$ and therefore $B^3 = 1$. So $B = 1$ which is impossible since in such case $A = 1$ also.

Now let K be the normal closure in M of the set

$$\{A^k, B^k, [A,B]^2, [A,[A,B]], [B,[A,B]]\}. \tag{16}$$

Clearly, M/K is a metabelian group of order $2k^2$ provided k is even. We claim that if in addition k is a multiple of 4 then K is a normal subgroup of Ω. Indeed, it is sufficient to show that for any element W of the set (16), $W^x, W^y \in K$. For $W = A^k$, $W^x = (A^k)^{-1} \in K$ and an easy computation shows that $W^y = (A^{-1}B)^k \equiv A^{-k}B^k[A,B]^{k(k-1)/2} \pmod K$. Therefore, if 4 divides k then $W^y \in K$. Similarly, one can deal with the remaining elements of (16).

It is easy to check that in $G = \Omega/K$, $[A,B]$ is conjugate to $(xy)^6$ whilst $[A,[A,B]]$ and $[B,[A,B]]$ are conjugate to $(xy)^6(yx)^6$. Therefore, G has the presentation

$$\langle x, y | x^2, y^3, (xy)^{12}, [x,y]^k, (xy)^6(yx)^6 \rangle \tag{17}$$

and xy represents there an element of order 12. Thus, G is a smooth Fuchsian factor group Δ/Γ of a Fuchsian group Δ with signature $[2,3,12]$ and so G acts as a group of automorphisms on a Riemann surface $X = \mathscr{H}/\Gamma$ which by the Hurwitz Riemann formula has the genus $g = (k^2/2) + 1$. Clearly the map $x \mapsto x^{-1}$, $y \mapsto y^{-1}$ induces an automorphism of G. So by a theorem of Singerman [22, Theorem 2], X is symmetric and therefore Γ is a normal subgroup of an NEC-group Λ with the signature $(2,3,7)$ as the last is the only one extending Δ. Let

$$\langle c_0, c_1, c_2 | c_0^2, c_1^2, c_2^2, (c_0c_1)^2, (c_1c_2)^3, (c_0c_2)^{12} \rangle$$

be the presentation of Λ and let a, b and c be the images of c_0, c_1 and c_2 respectively under the canonical projection from Λ onto $\widetilde{G} = \Lambda/\Gamma$. Observe that $x_1 = c_0c_1, x_2 = c_1c_2$ are the canonical generators of Δ and so we can assume that $x = ab$ and $y = bc$. Finally, $(ab)(ac)^6$ and $(ab)(bc)(ac)^6(cb)$ are elements having order 2 as both of them are conjugate to $(xy)^6x$. Therfore by (12), $|\theta(C(\Lambda, c_i))| = 8$ for $i = 0, 1$ and so by Theorem 3.2, X has $12(g-1)$ ovals. This completes the proof. □

However, it is worth to notice that $\widetilde{G}$ has the presentation

$$\langle a, b, c | a^2, b^2, c^2, (ab)^2, (bc)^3, (ac)^{12}, (acb)^{2k}, ((ac)^6b)^2 \rangle. \tag{18}$$

Indeed $x = ab$ and $y = bc$ generate in (18) a normal subgroup of index 2 having the presentation (17).

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